Singular Value Decomposition and its applications in Image Processing

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***Abstract*—Singular value decomposition (SVD) is an effective method for factoring matrices that is often a useful analysis tool. Although it can be used for a variety of purposes, a common application is data size reduction. The nature of SVD allows any size dataset to go through a reduction in rank with little loss of its innate variation. This ability and the insight that this analysis lends make it a popular tool for data and signal processing. Perhaps the best way to understand it, is to see how it works. In this paper we will discuss about the Singular Value Decomposition(SVD). We explain how to compute SVD for a given $m \times n$ matrix and its applications in image processing with examples.**

I. Introduction

Lemma 1. *[1] If $\Lambda = S^{-1}AS$ is a diagonal matrix then the diagonal elements of Λ are eigenvalues, and the columns of S are linearly independent eigenvectors of A.*

Lemma 2. *[1] If Λ is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2 \dots \lambda_r$ then its eigenvalues are its diagonal entries $\lambda_1, \lambda_2 \dots \lambda_r$.*

Lemma 3. *[1] For a matrix $A_{n\times n}$, if λ is a scalar and v is a nonzero vector such that $Av = \lambda v$, then λ is a eigenvalue of $A_{n\times n}$ and v is a corresponding eigenvector.*

Definition 1. *[1][2] The singular value decomposition of a matrix $A_{m\times n}$ is a factorization of $A_{m\times n}$ as $A = U\Sigma V^{\mathsf{T}}$ in which the matrices $U_{m\times r}, \Sigma_{r\times r}$ and $V_{n\times r}$ have the following properties:*

1. *$U_{m\times r}$ is a orthogonal matrix, $UU^{\mathsf{T}} = U^{\mathsf{T}}U = I$ which implies $U^{-1} = U^{\mathsf{T}}$.*
2. *$\Sigma_{r\times r}$ is a diagonal matrix whose diagonal entries $\sigma_1, \sigma_2 \dots \sigma_r$ are positive and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$.*
3. *$V_{n\times r}$ is a orthogonal matrix, $VV^{\mathsf{T}} = V^{\mathsf{T}}V = I$ which implies $V^{-1} = V^{\mathsf{T}}$.*

Given $A = U\Sigma V^{\mathsf{T}}$, Multiplying by A^{T} on both sides we get

$$\begin{aligned} A^{\mathsf{T}}A &= (U\Sigma V^{\mathsf{T}})^{\mathsf{T}}(U\Sigma V^{\mathsf{T}}) \\ &= (V\Sigma^{\mathsf{T}}U^{\mathsf{T}})(U\Sigma V^{\mathsf{T}}) \\ &= V\Sigma^{\mathsf{T}}\Sigma V^{\mathsf{T}} \ [\because U^{\mathsf{T}}U = I] \end{aligned}$$

where the diagonal entries of the diagonal matrix $\Sigma^{\mathsf{T}}\Sigma$ are $\sigma_1^2, \sigma_2^2 \dots \sigma_r^2$.

$$\begin{aligned} A^{\mathsf{T}}A &= V\Sigma^{\mathsf{T}}\Sigma V^{\mathsf{T}} \\ &= V\Sigma^{\mathsf{T}}\Sigma V^{-1} \ [\because V^{-1} = V^{\mathsf{T}}] \end{aligned}$$

By Lemma 1, we can say that the diagonal entries of diagonal matrix $\Sigma^{\mathsf{T}}\Sigma$ are the eigenvalues of $A^{\mathsf{T}}A$ which is a symmetric matrix and the columns of V are the eigenvectors of $A^{\mathsf{T}}A$. In a similar fashion, Multiplying A on both sides by A^{T} we get

$$\begin{aligned} AA^{\mathsf{T}} &= (U\Sigma V^{\mathsf{T}})(U\Sigma V^{\mathsf{T}})^{\mathsf{T}} \\ &= (U\Sigma V^{\mathsf{T}})(V\Sigma^{\mathsf{T}}U^{\mathsf{T}}) \\ &= U\Sigma\Sigma^{\mathsf{T}}U^{\mathsf{T}} \\ &= U\Sigma\Sigma^{\mathsf{T}}U^{-1} \end{aligned}$$

By Lemma 1, we can say that the diagonal entries of diagonal matrix $\Sigma^{\mathsf{T}}\Sigma$ are the eigenvalues of AA^{T} which is a symmetric matrix and the columns of U are the eigenvectors of AA^{T}.

Let $u_1, u_2 \dots u_r$ and $v_1, v_2 \dots v_r$ be the eigenvectors of $AA^{\mathsf{T}}, A^{\mathsf{T}}A$. We choose the eigenvectors to be orthonormal (since the eigenvectors of symmetric matrices $A^{\mathsf{T}}A, AA^{\mathsf{T}}$ are orthogonal). By Lemma 3 we have:

$$(A^{\mathsf{T}}A)v_i = \sigma_i^2 v_i \tag{1}$$

Multiplying by v_i^{T} on both sides

$$\begin{aligned} v_i^{\mathsf{T}}A^{\mathsf{T}}Av_i &= \sigma_i^2 v_i^{\mathsf{T}}v_i \\ v_i^{\mathsf{T}}A^{\mathsf{T}}Av_i &= \sigma_i^2 \ [\because \|v_i\|^2 = 1] \\ (Av_i)^{\mathsf{T}}(Av_i) &= \sigma_i^2 \\ \|Av_i\|^2 &= \sigma_i^2 \end{aligned}$$

$$\therefore \|Av_i\| = \sigma_i \tag{2}$$

Similarly,

$$AA^{\mathsf{T}}u_i = \sigma_i^2 u_i \tag{3}$$

Multiplying by A on both sides of (1)

$$(AA^{\mathsf{T}})(Av_i) = \sigma_i^2 Av_i \tag{4}$$

From (3), (4) we can say that the Av_i is a eigenvector of AA^{T} which is same u_i.

$$\therefore \frac{Av_i}{\sigma_i} = u_i \tag{5}$$

Since $\|Av_i\| = \sigma_i$ and u_i is a unit eigenvector.

II. Computing the SVD of a $m \times n$ matrix

Let us compute the SVD of a matrix

$$A_{2\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Computing the value of $A^\mathsf{T}A, AA^\mathsf{T}$

$$A^\mathsf{T}A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$$

$$AA^\mathsf{T} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$$

The charactersitic polynomial for $A^\mathsf{T}A$ is given by

$$\begin{aligned} |A^\mathsf{T}A - \lambda I| &= 0 \\ \implies \begin{vmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{vmatrix} &= 0 \\ \implies (10-\lambda)(40-\lambda) &= 0 \\ \implies 400 - 10\lambda - 40\lambda + \lambda^2 - 400 &= 0 \\ \implies \lambda^2 - 50\lambda &= 0 \\ \implies \lambda(\lambda - 50) &= 0 \\ \implies \lambda = 0,\ \lambda &= 50 \end{aligned}$$

To compute the eigenvector for the eigenvalue $\lambda = 0$ for $A^\mathsf{T}A$, we need to solve $(A^\mathsf{T}A - \lambda I)x = 0$

$$\begin{aligned} \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies 10x + 20y = 0,\ 20x + 40y &= 0 \\ \implies x + 2y &= 0 \\ \implies x &= -2y \end{aligned}$$

Let $y = t$ then $x = -2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenvector for the corresponding eigenvalue $\lambda = 0$ of $A^\mathsf{T}A$ is given by $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The unit eigenvector $v_1 = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$. Using (5) to find the value of u_1

$$\sigma_1 u_1 = Av_1 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

The unit eigenvector $u_1 = \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$ $[\because \sigma_1^2 = 0 \implies \sigma_1 = 0]$.

To compute the eigenvector for the eigenvalue $\lambda = 50$ for $A^\mathsf{T}A$, we need to solve $(A^\mathsf{T}A - \lambda I)x = 0$

$$\begin{aligned} \begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies -40x + 20y = 0,\ 20x - 10y &= 0 \\ \implies -2x + y &= 0 \\ \implies 2x &= y \end{aligned}$$

Let $x = t$ then $y = 2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenvector for the corresponding eigenvalue $\lambda = 50$ for $A^\mathsf{T}A$ is given by $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The unit eigenvector $v_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. Using (5) to find the value of u_2

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} \frac{5}{\sqrt{5}} \\ \frac{15}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

The unit eigenvector
$u_2 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$ $[\because \sigma_2^2 = 50 \implies \sigma_2 = 5\sqrt{2}]$

We can also directly obtain the u_i from AA^T.The characteristic polynomial for AA^T is given by

$$\begin{aligned} |AA^\mathsf{T} - \lambda I| &= 0 \\ \implies \begin{vmatrix} 5-\lambda & 15 \\ 15 & 45-\lambda \end{vmatrix} &= 0 \\ \implies (5-\lambda)(45-\lambda) - 225 &= 0 \\ \implies 225 - 5\lambda - 45\lambda + \lambda^2 - 225 &= 0 \\ \implies \lambda^2 - 50 &= 0 \\ \implies \lambda(\lambda - 50) &= 0 \\ \implies \lambda = 0,\ \lambda &= 50 \end{aligned}$$

To compute the eigenvector for the eigenvalue $\lambda = 0$ for AA^T, we need to solve $(AA^\mathsf{T} - \lambda I)x = 0$

$$\begin{aligned} \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies 5x + 15y = 0,\ 15x + 45y &= 0 \\ \implies x + 3y &= 0 \\ \implies x &= -3y \end{aligned}$$

Let $y = t$ then $x = -3t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The eigenvector for the corresponding eigenvalue $\lambda = 0$ of AA^T is $u_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. The unit eigenvector $u_1 = \begin{bmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$

To compute the eigenvector for the eigenvalue $\lambda = 50$ for AA^T, we need to solve $(AA^\mathsf{T} - \lambda I)x = 0$

$$\begin{bmatrix}-45 & 15\\ 15 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\implies -45x + 15y = 0,\ 15x - 5y = 0$$
$$\implies -3x + y = 0$$
$$\implies 3x = y$$

Let $x = t$ then $y = 3t$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}$$

The eigenvector for the corresponding eigenvalue $\lambda = 50$ of $AA^\top$ is $u_2 = \begin{bmatrix}1\\ 3\end{bmatrix}$. The unit eigenvector $u_2 = \begin{bmatrix}\frac{1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{bmatrix}$

We have

$$A = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}, V = \begin{bmatrix}\frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}$$

$$U = \begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}, \Sigma = \begin{bmatrix}0 & 0\\ 0 & 5\sqrt{2}\end{bmatrix}$$

Verification of $A = U\Sigma V^\top$

$$A = U\Sigma V^\top = \begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 5\sqrt{2}\end{bmatrix}\begin{bmatrix}\frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}^\top$$
$$= \begin{bmatrix}0 & \frac{5\sqrt{2}}{\sqrt{10}}\\ 0 & \frac{15\sqrt{2}}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{5\sqrt{2}}{\sqrt{50}} & \frac{10\sqrt{2}}{\sqrt{50}}\\ \frac{15\sqrt{2}}{\sqrt{50}} & \frac{30\sqrt{2}}{\sqrt{50}}\end{bmatrix}$$
$$= \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$

III. Rank-k Approximation from the SVD

[5] The SVD of a matrix A can be used to approximate A by a rank-k matrix where $k \geq 1$. The rank-k approximation of a matrix A can be computed as follows:

1. Compute the SVD of the $A_{m\times n} = U_{m\times r}\Sigma_{r\times r}{V^\top}_{r\times n}$.
2. Keep the left k column vectors of $U_{m\times r}$ so that $U_{m\times r}$ becomes $U_{m\times k}$.
3. Keep k row, column vectors of $\Sigma_{r\times r}$ so that $\Sigma_{r\times r}$ becomes $\Sigma_{k\times k}$.
4. Keep the top k row vectors of ${V^\top}_{r\times n}$ so that $V^\top_{r\times n}$ becomes ${V^\top}_{k\times n}$
5. Compute the rank-k approximation $A_k = U_{m\times k}\Sigma_{k\times k}{V^\top}_{k\times n}$ where A_k is a $m\times n$ matrix.

The error in the approximation is given by the Frobenius norm $\|A - A_k\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} {a_{ij}}^2}$

A. Example

Let $A = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$, the SVD of $A = U\Sigma V^\top$ where

$$V = \begin{bmatrix}\frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix} U = \begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}, \Sigma = \begin{bmatrix}0 & 0\\ 0 & 5\sqrt{2}\end{bmatrix}$$

The rank-1 approximation of A, $A_1 = U_1\Sigma_1{V_1}^\top$ where

$$U_1 = \begin{bmatrix}\frac{-3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{bmatrix}\ \Sigma_1 = \begin{bmatrix}5\sqrt{2}\end{bmatrix}\ V_1 = \begin{bmatrix}\frac{-2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix}$$

$$A_1 = \begin{bmatrix}\frac{-3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}5\sqrt{2}\end{bmatrix}\begin{bmatrix}\frac{-2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix}^\top$$
$$= \begin{bmatrix}\frac{-15\sqrt{2}}{\sqrt{10}}\\ \frac{5\sqrt{2}}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{30\sqrt{2}}{5\sqrt{2}} & \frac{-15\sqrt{2}}{5\sqrt{2}}\\ \frac{-10\sqrt{2}}{5\sqrt{2}} & \frac{5\sqrt{2}}{5\sqrt{2}}\end{bmatrix}$$
$$= \begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$$

$$A - A_1 = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix} - \begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$$
$$= \begin{bmatrix}-5 & 5\\ 5 & 5\end{bmatrix}$$

The Frobenius norm [8] $\|A - A_1\|_F$ is given by

$$\|A - A_1\|_F = \sqrt{(-5)^2 + 5^2 + 5^2 + 5^2} = \sqrt{100} = 10$$

IV. Applications of SVD in Image Processing

A. Image Overview

A color image consists of $m \times n$ pixels where each pixel is made up of Red, Green and Blue channels. The image can be represented as a $m \times n$ matrix for each of the Red, Green and Blue channels. We use the Lenna test image [7] as the example, shown below is the test image along with its Red, Green and Blue channels

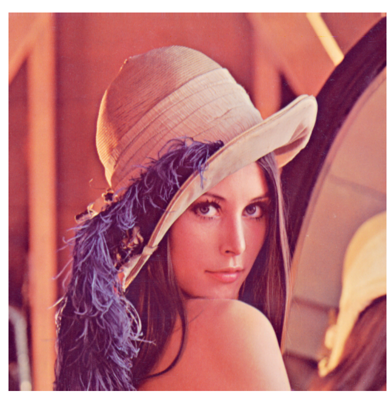

Fig. 1: Lenna test image 512×512 pixels

Fig. 2: Red channel

Fig. 3: Green channel

Fig. 4: Blue channel

B. *Image Compression*

An image I requires $m \times n \times 3$ bytes for storage, the motivation behind image compression is to represent I using less number of bytes. Since I can be represented as a $m \times n$ matrix for each channel. We can compute the rank-k approximation of each of the channels using the SVD and combine them to get the image $I^{'}$. The compression percentage of the image is given by [3]

$$R = \frac{(mk + nk + k) * 100}{mn}$$

Fig. 5: rank-8 approximation, $R = 3.128$

Fig. 6: rank-16 approximation, $R = 6.256$

Fig. 7: rank-32 approximation, $R = 12.512$

V. Conclusion

In this paper we have discussed SVD, rank-k approximation from the SVD and its application in image processing i.e. image compression. There are other applications of SVD like Image Denoising, Image Watermarking, Image reconstruction. We would like to refer the reader to [3] [5] [6] for theory and applications of the SVD.

Fig. 8: rank-64 approximation, $R = 25.024$

Fig. 9: rank-128 approximation, $R = 50.048$

Fig. 10: rank-256 approximation, $R = 100.097$

Appendix A
Matlab Code for rank-k approximation of the image from the SVD

```matlab
A = imread('Lenna_test.png');
A = im2double(A);
R = A(:,:,1);
G = A(:,:,2);
B = A(:,:,3);

zero_channel = zeros(size(A, 1), size(A, 2));

red_img = cat(3, R, zero_channel, zero_channel);
green_img = cat(3, zero_channel, G, zero_channel);
```

Fig. 11: rank-512 approximation, $R = 200.195$

```matlab
blue_img = cat(3, zero_channel, zero_channel, B);
imwrite(red_img, 'Lenna_red.png', 'png');
imwrite(green_img, 'Lenna_green.png', 'png');
imwrite(blue_img, 'Lenna_blue.png', 'png');
% Show the Red, Green, Blue channels of the image
% imshow(im2uint8(red_img));
% imshow(im2uint8(green_img));
% imshow(im2uint8(blue_img));

[u_red, s_red, v_red] = svd(R);
[u_green, s_green, v_green] = svd(G);
[u_blue, s_blue, v_blue] = svd(B);


for k = [8, 16, 32, 64, 128, 256, 512]
    % A = U S V' where V' is the transpose of V
    % computing the low rank approximation of each component i.e R, G, B
    Ak_red = u_red(:,1:k) * s_red(1:k,1:k) * v_red(:, 1:k)';
    Ak_green = u_green(:,1:k) * s_green(1:k,1:k) * v_green(:, 1:k)';
    Ak_blue = u_blue(:, 1:k) * s_blue(1:k, 1:k) * v_blue(:, 1:k)';
    img_k = cat(3, Ak_red, Ak_green, Ak_blue);
    imwrite(img_k, strcat('Lenna_test_', num2str(k), '.png'), 'png');
end
```

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