EN 535.441 Discussion Activity 2: Application of Laplace Transform

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1 Introduction

In this report, we show how Laplace transforms can be used to obtain a solution for the set of differential equations describing projectile motion. The position of a projectile in motion can be represented as a parametric set of differential equations describing the position in the horizontal and the vertical dimension. These differential equations are independent ordinary differential equations that can be solved using the Laplace transform.

2 Model and Equation Set Up

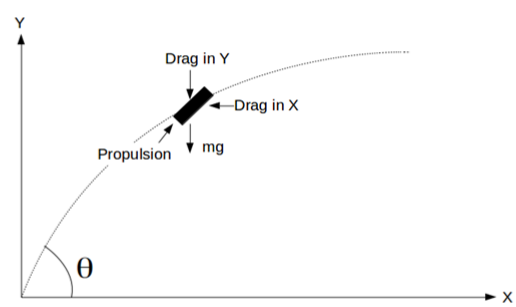

Figure 1: Projectile in Motion

In this model, we define the vertical direction as y and the horizontal direction as x. The projectile is assumed to be relatively close to the Earth such that the force due to Earth's gravity can be represented as mg where m is the mass of the projectile and g is the acceleration due to Earth's gravity. At this point, the parametric differential equations are:

$$X : m\frac{d^2x}{dt^2} = 0 \tag{1}$$

$$Y : m\frac{d^2y}{dt^2} = -mg \tag{2}$$

At heights close to the surface of the Earth, the air resistance becomes a factor. This can be represented as a constant multiplied by the magnitude of the velocity in that direction (simplified for the purposes of calculating the Laplace transform). This force always acts opposite to the direction of the velocity. The parametric equations become:

$$X : m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \tag{3}$$

$$Y : m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt} \tag{4}$$

An additional term can be added if the projectile provides a force on itself from propulsion. If we assume the mass of the projectile remains constant as fuel is burned for propulsion, the propulsion force p can be represented as a step function where the propulsion occurs for an allotted amount of time t_p. Here we also assume the propulsion force acts at an angle based on the initial trajectory. The parametric equations now become:

$$X : m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} + p(1 - u(t - t_p))\cos\theta \tag{5}$$

$$Y : m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt} + p(1 - u(t - t_p))\sin\theta \tag{6}$$

where the angle θ is the angle from horizontal at time equals 0. We can solve these two differential equations using Laplace transforms to develop a set of parametric equations for the motion of the projectile based on the parameters mentioned, where the initial conditions of positions in x and y and velocities in x and y are 0 at time 0.

3 Solution for x

$$mx'' = -kx' + p\cos\theta(1 - u(t - t_p)),\ x(0) = 0,\ x'(0) = 0 \tag{7}$$

Applying laplace transform on both sides of (7)

$$m\mathcal{L}(x'') = -k\mathcal{L}(x') + p\cos\theta[\mathcal{L}(1) - \mathcal{L}(u(t - t_p))]$$

$$m(s^2X - SX(0) - X'(0)) = -k(SX - X(0)) + p\cos\theta(\frac{1}{s} - \frac{e^{-t_ps}}{s})$$

$$X = \frac{p\cos\theta}{s^2(ms + k)}(1 - e^{-t_ps})$$

Applying inverse laplace transform on both sides

$$x(t) = \mathcal{L}^{-1}(\frac{p\cos\theta(1-e^{-t_p s}}{s^2(ms+k)})$$

Using partial fractions

$$\frac{1}{s^2(ms+k)} = \frac{A}{ms+k} + \frac{B}{s} + \frac{C}{s^2}$$

$$= \frac{A}{ms+k} + \frac{Bs+C}{s^2}$$

$$= \frac{s^2(A+Bm)+s(Bk+cm)+ck}{s^2(ms+k)}$$

Multiplying by $s^2(ms+k)$ on both sides

$$1 = s^2(A+Bm)+s(Bk+cm)+ck$$

Equating the coefficients of s on both sides we get

$$A+Bm=0, Bk+Cm=0, Ck=1$$

$$C=\frac{1}{k},\ A=\frac{m^2}{k^2}\ B=\frac{-m}{k^2}$$

$$\frac{1}{s^2(ms+k)} = \frac{m^2}{k^2(ms+k)} + \frac{-m}{sk^2} + \frac{1}{s^2k} \tag{8}$$

Substituting the partial fraction expansion (8) in $x(t)$

$$x(t) = (p\cos\theta)\mathcal{L}^{-1}(\frac{1}{s^2(ms+k)} - \frac{e^{-t_p s}}{s^2(-ms+k)})$$

$$x(t) = (p\cos\theta)\mathcal{L}^{-1}(\frac{m^2}{k^2(ms+k)} + \frac{-ms+k}{s^2k^2} - e^{-t_p s}(\frac{m^2}{k^2(ms+k)} + \frac{(-ms+k)}{s^2}))$$

$$x(t) = (p\cos\theta)(\frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s+\frac{k}{m}}) - \frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s}) + \frac{1}{k}\mathcal{L}^{-1}(\frac{1}{s^2}) - \frac{m}{k^2}\mathcal{L}^{-1}(\frac{e^{-t_p s}}{s+\frac{k}{m}}) + \frac{m}{k^2}\mathcal{L}^{-1}(\frac{e^{-t_p s}}{s}) - \frac{1}{k}\mathcal{L}^{-1}(\frac{e^{-t_p s}}{s^2}))$$

Using the Second shifting theorem: If $F(s) = \mathcal{L}(f)$, then $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)$

$$x(t) = (p\cos\theta)(\frac{m}{k^2}e^{-\frac{kt}{m}} - \frac{m}{k^2} + \frac{t}{k} - \frac{m}{k^2}e^{-\frac{k(t-t_p)}{m}}u(t-t_p) + \frac{m}{k^2}u(t-t_p) - \frac{u(t-t_p)(t-t_p)}{k})$$

$$x(t) = (p\cos\theta)(\frac{m}{k^2}e^{-\frac{kt}{m}} - \frac{m}{k^2} + \frac{t}{k} - u(t-t_p)(\frac{m}{k^2}e^{-\frac{k(t-t_p)}{m}} - \frac{m}{k^2} + \frac{(t-t_p)}{k}))$$

4 Solution for y

$$my'' = -mg - ky' + p\sin\theta(1 - u(t-t_p)), y(0) = 0, y'(0) = 0 \tag{9}$$

Applying laplace transform on both sides of (9)

$$m\mathcal{L}(y'') = -mg\mathcal{L}(1) - k\mathcal{L}(y') + p\sin\theta(\mathcal{L}(1) - \mathcal{L}(u(t-t_p)))$$

$$m(s^2Y - SY(0) - Y'(0)) = -\frac{mg}{s} - k(sY - sY(0)) + p\sin\theta\frac{(1-e^{-t_ps}}{s})$$

$$Y = \frac{-mg + p\sin\theta(1-e^{-t_ps})}{s^2(ms+k)}$$

Applying the inverse laplace transform on both sides

$$y(t) = \mathcal{L}^{-1}(\frac{-mg + p\sin\theta(1-e^{-t_ps})}{s^2(ms+k)})$$

Substituting the partial fraction expansion (8) in $y(t)$

$$y(t) = -mg(\mathcal{L}^{-1}(\frac{m^2}{k^2(ms+k)} + \frac{-ms+k}{s^2k^2}))$$
$$+(p\sin\theta)\mathcal{L}^{-1}(\frac{m^2}{k^2(ms+k)} + \frac{-ms+k}{s^2k^2} - e^{-t_ps}(\frac{m^2}{k^2(ms+k)} + \frac{(-ms+k)}{s^2}))$$

$$y(t) = -mg(\frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s+\frac{k}{m}}) - \frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s}) + \frac{1}{k}\mathcal{L}^{-1}(\frac{1}{s^2}))$$
$$+ (p\sin\theta)(\frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s+\frac{k}{m}}) - \frac{m}{k^2}\mathcal{L}^{-1}(\frac{1}{s}) + \frac{1}{k}\mathcal{L}^{-1}(\frac{1}{s^2}) - \frac{m}{k^2}\mathcal{L}^{-1}(\frac{e^{-t_ps}}{s+\frac{k}{m}})$$
$$+ \frac{m}{k^2}\mathcal{L}^{-1}(\frac{e^{-t_ps}}{s}) - \frac{1}{k}\mathcal{L}^{-1}(\frac{e^{-t_ps}}{s^2}))$$

Using the Second shifting theorem: If $F(s) = \mathcal{L}(f)$, then $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)u(t-a)$

$$y(t) = -mg(\frac{m}{k^2}e^{-\frac{kt}{m}} - \frac{m}{k^2} + \frac{t}{k})$$
$$+ (p\sin\theta)(\frac{m}{k^2}e^{-\frac{kt}{m}} - \frac{m}{k^2} + \frac{t}{k} - u(t-t_p)(\frac{m}{k^2}e^{-\frac{k(t-t_p)}{m}} - \frac{m}{k^2} + \frac{(t-t_p)}{k}))$$

5 Observations

An interesting exercise to perform with these equations would be to substitute values and compare to expected behavior. For example, in the case of no propulsion time ($t_p = 0$), the two terms in the x direction directly cancel and $x(t) = 0$. The propulsion term in the y direction also cancels and only the gravitational term remains. This can also be achieved by assuming the force due to propulsion $p = 0$.

Another example would be the case with zero air resistance ($k = 0$). In this case, the original differential equations lose the air resistance term and the parametric equations become:

$$x(t) = \frac{p\cos\theta}{2m}(t^2 - u(t-t_p)(t-t_p)^2)$$
$$y(t) = \frac{p\sin\theta}{2m}(t^2 - u(t-t_p)(t-t_p)^2) - \frac{gt^2}{2}$$

This results in the classical 2nd order equation for $y(t)$ where $y(t) = -\frac{gt^2}{2}$ if $t_p = 0$. To obtain the classical equations for motion, one can include nonzero values for the initial positions and velocities. This results in the following parametric equations:

$$x(t) = \frac{p\cos\theta}{2m}(t^2 - u(t-t_p)(t-t_p)^2) + x'(0)t + x(0)$$
$$y(t) = \frac{p\sin\theta}{2m}(t^2 - u(t-t_p)(t-t_p)^2) - \frac{gt^2}{2} + y'(0)t + y(0)$$

6 Further Work

Several additions can be made to this model in order to improve the accuracy including:

- Variable Mass: In reality, the mass of the projectile would change as fuel for propulsion is used. This could be included by substituting a function of time for the constant mass.

- Variable Air Density: As the projectile reaches significant height, the force due to air resistance becomes less as the air density decreases. This could be represented by changing the air resistance constant to be a function of y. The equation for the force due to air resistance could also be made more accurate by squaring the first derivative as the force due to air resistance is proportional to velocity squared.
- Variable Gravity: The force due to gravity also varies as the projectile reaches significant heights. This can also be represented by changing the gravity constant to be a function of y.

References

[1] Rocket Trajectory, Systems and Labs, Fall 2008